Topological string on genus one fibered CY 3-folds with N-sections and Jacobi-Forms

Online China Geometry and Physics Seminar, Zhejing 12. May 2020

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Motivation:

Elliptically fibered Calabi-Yau 3-folds construct non-perturbative 6d vacua via F-theory.

- The topological string partition function Z counts the BPS spectrum of theses theories

- In the decoupling limit to 6d super conformal field theories Z reproduces the elliptic genus of non-critical string theories

In this talk we want to:

- Construct this topological string partition function Z from the special symmetries of elliptic CY- 3 folds with N-sections

- Study its decoupling limit using a novel 6d version of Nakajima's and Yoshika's blow up equations

The topological string partition:

Perturbative string theory is defined by weighted maps

$$\Phi : \Sigma_g \to M(\Omega, \omega) \times \mathbb{R}_{3,1}$$

from a 2d genus g world-sheet Σ_g into a CY-3-fold target space. Here

- ω is the $(1,1)$-Kähler form of M, $t_\beta = \int_{C_\beta} \omega + ib$ the Kähler structure
- $\Omega(z)$ is the $(3,0)$ form, depending on the complex structure parametrized by the periods $X^I = \int_{A^I} \Omega(z)$

A-model twist yields a localisation of the topological string partition function holomorphic maps hence

$$Z_A^{hol}(\omega, \lambda) = \exp\left(\sum_{\kappa \in H_2(M,\mathbb{Z})} \sum_{g=0}^{\infty} r_g^{\kappa} \lambda^{2g-2} Q^{\kappa} \right)$$

Here $Q^{\kappa} = e^{-t_{\beta}}$, λ the string coupling and $r_g^{\beta} \in \mathbb{Q}$ the Gromov-Witten invariants. Generally the free energy is

$$F = \log(Z) = \sum_g \lambda^{2g-2} F_g$$

$Z_A^{hol}(\omega, \lambda)$ asymptotic in λ, but F_g convergent in Q!

B-model twist yields a localisation to constant maps with the BCOV measure. One expects ‘roughly‘

$$Z_B(\Omega, W) \ \ \text{‘}=\text{‘} \ \ Z_A(\omega, M)$$

for mirror pairs of manifolds $h_{k,k}(M) = h_{3-k,k}(W)$, $k = 0, \ldots, 3$.

Roughly means: taking a holomorphic limit of $Z_B(\Omega, W)$ at a point of maximal unipotent monodromy one gets $\mathcal{Z}_A^{hol}(\omega, \lambda)$. However one expects that a global $Z_A(\omega, M)$ exists in the quantum geometry of M, with exactly the properties defined in the B-model below.

$Z_B(\Omega, W)$ depends on the periods

$$X^I = \int_{A^I} \Omega, \qquad F_I = \int_{B_I} \Omega$$

where $A^I \cap B_J = \delta^I_J$ is an integer symplectic basis of $H_3(W, \mathbb{Z})$. In particular with $t^i = X^i / X_0(z)$ one gets

$$F_0^{hol}(t) = \frac{1}{(X^0)^2} X^I F_I(z(t))$$

One important structure in the B model are symplectic transformations

$$\begin{pmatrix} F_I \\ X^I \end{pmatrix} \to \mathbf{S} \begin{pmatrix} F_I \\ X^I \end{pmatrix},$$

with $\mathbf{S} \in \mathrm{Sp}(h_{21}+1, \mathbb{R})$, which raise the questions:

- Q1.) How does Z transform under $\mathrm{Sp}(h_{21}+1, \mathbb{R})$ transformations?

- Q2.) What does the invariance of Z under those $\mathrm{Sp}(h_{21}+1, \mathbb{Z})$ transformations imply, that generate the monodromy group Γ_W of the complex family?

A1.) Witten '93 interprets $\Psi(X^I) = \langle X^I, Z\rangle = Z(X^I)$ as wave function on $H_3(M,\mathbb{R})$. Changing the polarisation is equivalent to changing the complex structure. The infinitesimal Bogoliubov transformation has been found to be different and as Huang, Katz, AK '20

$$\left[\bar\partial_{\bar\imath} - \left(\frac{\chi}{24} - 1\right) K_{\bar\imath} - C_{\bar\imath}^{ij}\left(\frac{\lambda^2}{2} D_i D_j - D_i F_0 D_j\right)\right]\Psi = 0 \quad (1)$$

$C_{\bar k}^{ij} = e^{2K} G^{i\bar\imath} G^{j\bar\jmath} \bar C_{\bar\imath\bar\jmath\bar k}$, the Kähler potential K with WP metric $G_{i\bar\jmath}$, the couplings C_{ijk} and the covariant D_i are given by special geometry in terms of the periods.

The following differences make (1) equivalent to the BCOV holomorphic anomaly equation:

- $\Psi \in \mathcal{L}^{\frac{\chi}{24}-1}$ is a on trivial section of the Kähler line bundle to reproduce the genus one HAE

- F_0 is the non-holomorphic prepotential defined by $C_{ijk} = -D_i D_j D_k F_0$, constructed from BCOV propagators

$$\begin{pmatrix} 2S & S_{\bar{a}} \\ S_{\bar{b}} & S_{\bar{a}\bar{b}} \end{pmatrix} = e^{2K} \bar{D}_{\bar{\alpha}} \bar{X}^I (F_{IJ} - \bar{F}_{IJ}) \bar{D}_{\bar{\beta}} \bar{X}^J$$

as $F_0 = -e^{-2K} \bar{S} = -\frac{1}{2} X^I (F_{IJ} - \bar{F}_{IJ}) X^J$ (*).

Indeed if we plug $\Psi = \exp\left(\sum_g \lambda^{2g-2} F_g\right)$ into (1) one gets : at order λ^{-2}, $\bar{\partial}_{\bar{\imath}} F_0 = -\frac{1}{2} C^{ij}_{\bar{\imath}} D_i F_0 D_j F_0$ true by (*), taking the D_a derivative at order λ^0 yields the genus one HAE and at λ^{2g-2}, $g > 1$ yields the remaining HAE's.

A2.) BCOV 93' and YY '04,... : propagators and holomorphic generators $X^{\alpha\beta}$ form a ring of "quasi modular" forms of Γ_W, so that $F_g(S, S_a, S_{ab}, X^{\alpha\beta})$ a weighted $(3,2,1,0)$ homogeneous polynomial of degree $3g-3$. Holomorphic kernel of (1) can be fixed to some extend by modularity and boundary conditions.

Today we want to use the monodromy properties of (the mirrors of) elliptic Calabi-Yau manifolds and A1.) and A2.) to derive properties of $Z_B(\Omega)$.

Let M be an elliptically fibred 3-fold over a 2d surface B

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & M \\ & & \downarrow \pi \\ & & B \end{array}$$

I) Divisor classes:

- a.) Sections of π that are independent in the Mordell-

Weil Group: E_0 denotes either a zero section or an N-section, and E_i, $i = 1, \ldots, r$ rational section $\rightarrow$ $U(1)^r$ abelian gauge symmetry.

- b.) fibral divisors D_i^f rational curves fibred over $\tilde{D}_k^{sing} \subset B$. The come from resolving Kodaira singularities $\rightarrow$ non-abelian gauge symmetries G^{na}

- c.) vertical divisors $\pi^{-1}(\tilde{D}_i) = D_i$, $\tilde{D}_i \subset B$, $i = 1, \ldots, b_2(B)$.

We denote the Kähler parameters in the expansion of Kähler form in terms of cohomology classes (name as the dual divisors) as

$$\omega = \tau \cdot D_\tau + \sum_{i=1}^{r} m_i \cdot \sigma(E_i) + \sum_{i=r+1}^{\mathsf{rk}(G)} m_i \cdot D^f_{i-r} + \sum_{i=1}^{b_2(B)} \tilde{t}_i \cdot D'_i$$

Here τ volume of the elliptic curve (E), $D_\tau = E_0 + D$, $\sigma(E_i)$ the Tate-Shioda map, $G = G^{na} \times U(1)^r$ and D'_i dual to $\frac{1}{N}E_0 \cdot D_i$.

II) Some elements of categorical mirror symmetry:

Symmetries:

Monodromies on periods of W	$\longleftrightarrow$	Autoequivalences in $D^b(M)$ derived category of quasi − coherent sheafs $\mathcal{F}$

Central Charges:

$\Pi_\Gamma = \int_\Gamma \Omega$ $\Gamma \in H_3(W, \mathbb{Z})$	$\longleftrightarrow$	$\Pi(\mathcal{F}^*)$ $\Pi_{asy} = \int_M e^\omega \Gamma_{\mathbb{C}}(T_M) ch(\mathcal{F}^*)^\vee$

Pairing:

$$\begin{array}{l} \text{Intersect. } \Gamma_k \cap \Gamma_l \\ \Gamma_k, \Gamma_l \in H_3(W, \mathbb{Z}) \end{array} \leftrightarrow \begin{array}{l} \chi(\mathcal{E}^*, \mathcal{F}^*) = \\ \int_M \mathrm{Td}(T_M) ch(\mathcal{E}^*) ch(\mathcal{F}^*)^\vee \end{array}$$

Symmetry generators:

$$\begin{array}{lcl} \text{Monodromy :} & & \text{Fourier Mukai Transform :} \\ M : \Gamma \to M_\gamma \Gamma & \leftrightarrow & \Phi_{\mathcal{E}} :\to R\pi_{1*}(\mathcal{E} \otimes_L L\pi_2^* \mathcal{F}^*) \\ \gamma \text{ loop in } \mathcal{M}_{cs}(W) & & \text{applied to diag } \Delta M \subset M \times M \end{array}$$

$$\begin{array}{lcl} \text{vanishing cycles } \nu & & \text{Fourier Mukai kernel } \mathcal{E} \\ M : \Pi(\Gamma) \mapsto \Pi(\Gamma) & \leftrightarrow & \Pi(\mathcal{F}) \mapsto \Pi(\mathcal{F}^*) \\ -(\Gamma \cap \nu)\Pi(\nu) & & -\chi(\mathcal{F}^*, \mathcal{E})\Pi(\mathcal{E}) \end{array}$$

Two distinguished monodromies on elliptic fibrations:

Seidel '98,..., Katz, Huang, Klemm, '16, Schimannek '19, that extend the T, $T : \mathcal{F}^* \mapsto \mathcal{F}^* \otimes \mathcal{O}(D_\tau)$ and the U $(= ST)$, FMK $\nu = \Theta_{\mathcal{E}}$, transformations of $\Gamma_1(N)$ to the classes of the elliptically fibred Calabi-Yau manifolds.

Let $Q_i = \exp(2\pi\tilde{t}_i + \frac{\tilde{a}_i}{2N}\tau)$, $\tilde{a}_i = \int_M D_\tau^2 D_i$, $a_i = \int_B c_1(M)\tilde{D}_i$ then the parameters transform

$$\color{blue}{T} : \begin{cases} \tau \mapsto \tau + 1 \\ m_i \mapsto m_i \\ Q_i \mapsto (-1)^{\frac{\tilde{a}_i}{2N}} Q_i \end{cases}$$

and

$$\color{red}{U} : \begin{cases} \tau & \mapsto & \tau/(1+N\tau) \\ m_i & \mapsto & m_i/(1+N\tau)\,, \quad i = 1, \ldots, \mathrm{rk}(G) \\ Q_i & \mapsto & (-1)^{a_i} \exp\left(-\frac{1}{1+N\tau} \cdot \frac{1}{2} m^a m^b C^i_{ab} + \mathcal{O}(Q_i)\right) Q_i \end{cases}$$

For singular or elliptic fibrations with many sections one has also shifts $M_a : m_i \to m_a + 1$ by tensoring with $\mathcal{O}(D_a^f)$ and $\mathcal{O}(\sigma(E_a))$ respectively.
$E_a := M_i \cdot U^{-1} \cdot M_a^{-1} \cdot U$ acts as

$$E_a : \begin{cases} \tau & \mapsto & \tau \\ m_i & \mapsto & m_i \, , \quad i = 1, ..., \mathsf{rk}(G), \, i \neq a \\ m_a & \mapsto & m_a + N \cdot \tau \, , \\ Q_i & \mapsto & \exp\left(\frac{N \cdot \tau}{2} C^i_{aa} + C^i_{(ab)} m^b \right) Q_i \end{cases}$$

Consequences of the symmetries:

Expand the Z in terms of the base degrees class $\beta \in H_2(B, Z)$ as

$$Z(\underline{t}, \tau, \underline{m}, \lambda) = Z_0(\tau, \underline{m}, \lambda)\left(1 + \sum_{\beta \in H_2(B,\mathbb{Z})} Z_\beta(\tau, \underline{m}, \lambda) Q^\beta\right) .$$

Then $Z_{\beta>0}$ is a meromorphic Jacobi form of $\Gamma_1(N)$ with modular parameter τ and elliptic parameters $\underline{m}, \lambda$ of weight $k = 0$.

This follows from the transformations of Z under Γ_W

and all Jacobi form indices are fixed by intersection calculation on M.

The second degree index polynomial in the elliptic parameters $\underline{m}$ is given in terms of the quadratic intersection

$$C^i_{ab} = \begin{cases} -\pi_*\left(\sigma(E_a)\cdot\sigma(E_b)\right)\cdot C_\beta & \text{for } 1\le a,b\le r \\ -\pi_*\left(D_{f,a}\cdot D_{f,b}\right)\cdot C_\beta & \text{for } r< a,b\le \mathrm{rk}(G) \\ 0 & \text{otherwise} \end{cases}$$

as

$$m^{\underline{m}} = \frac{1}{2N} m^a m^b C^i_{ab} \ ,$$

The corresponding Jacobi transformation property follows from the invariance of $Z_\beta(\tau, \underline{m}, \lambda)Q^\beta$ under the M_i and E_i monodromies, i.e. the invariance of Z under transformations which do not change the polarization (no exchange $X^I \leftrightarrow P_I = F_I \sim \partial_{X^I} F$). The first line, i.e. the case of rational sections with no N-section was observed Lee,Lerche,Weigand '18.

The index w.r.t λ follows from (1) and in a slight modification of the argument of HKK'15 as

$$m^\lambda = \frac{1}{2N}\beta \cdot (\beta - c_1(B)) \ .$$

Note that if the base has a contractible configurations of curves and gravity can be decoupled in $6d$, the information of these indices is equivalent to the anomaly polynomial of the $(1,0)$ 6d theory by the indentification del Zotto, Gu, Lockhart, Kashani-Poor,AK '17.

$$\begin{matrix}\text{intersection} \\ \text{form on } M\end{matrix} \leftrightarrow \begin{matrix}\text{Index of} \\ \text{Jacobi form}\end{matrix} \leftrightarrow \begin{matrix}\text{6d } (1,0) \text{ anomaly} \\ \text{polynom}\end{matrix}$$

Jacobi form ansatz of the answer

The meromorphic Jacobi-form have a numerator which is fixed by poles of the Gopakumar Vafa expansion

$$Z_\beta(\tau, \underline{m}, \lambda) = \frac{1}{\eta(\tau)^{12\cdot c_1(B)\cdot\beta}} \frac{\phi_\beta(\tau, \underline{m}, \lambda)}{\prod_{l=1}^{b_2(B)} \prod_{s=1}^{\beta_l} \phi_{-2,1}(\tau, s\lambda)}, \tag{2}$$

so that the numerator is a weak Jacobi-Form of $\Gamma_1(N)$. The ring of modular forms for $\Gamma_1(N)$ and

$N \in \{1, 2, 3, 4\}$ are generated by

$$\begin{aligned} M_*(1) =& \langle E_4(\tau),\ E_6(\tau) \rangle \,, \\ M_*(2) =& \langle E_2^{(2)}(\tau),\ E_4(\tau) \rangle \,, \\ M_*(3) =& \langle E_2^{(3)}(\tau),\ E_4(\tau),\ E_6(\tau) \rangle \,, \\ M_*(4) =& \langle E_2^{(2)}(\tau),\ E_2^{(4)}(\tau),\ E_4(\tau),\ E_6(\tau) \rangle \,. \end{aligned} \tag{3}$$

With this information the ansatz can be fixed for the local cases completely and for the global cases to large extend.

Type II/CHL duality and SQFT's as applications:

Let B be a rationally fibred surface, e.g. $\mathbb{F}_1$ with fibre $F^2 = 0$ and section $S^2 = -1$. M has I.) a $K3$ fibration phase and II.) an elliptic fibration phase with a -1 curve in the base

$$\begin{array}{ccccc} \mathcal{E} & \longrightarrow & K3 & \longrightarrow & M \\ & & & & \big\downarrow \pi_1 \\ & & & & \mathbb{P}^1_{S'} \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{E} & \longrightarrow & \frac{1}{2}K3 \\ & & \big\downarrow \pi_2 \\ & & \mathbb{P}^1_{S} \end{array}$$

- I.) Using fiberwiseType IIA/heterotic duality we can ask what the heterotic dual to N-section geometry is?

- II.) What are the local theories associated to the $\frac{1}{2}K3$?

AI) A wide class of $(K3 \times T^2)/\mathbb{Z}_N$ heterotic CHL model have been constructed in connection with Matthieu moonshine and umbral moonshine. In particular using the Borcherds lift the heterotic one loop amplitude of Antoniadis, Gava Narain and Tayler that yields

$$\int F_g(t,\bar{t}) F_+^{2g-2} R_+^2$$

have been for some theories by Chattopadhyaya and

David '17 and '18. Based on comparison of the higher genus calculation we have now exactly identified the N-section Calabi-Yau duals and found many more type IIA candidtates for heterotic CHL duals.

AII.) For all these local models topological string can be refined. $\lambda \to (\epsilon_1, \epsilon_2)$ and the ansatz generalises Gu, Huang, Kashani-Poor, Klemm '17.

The answer for the 7 cases $N = 1$ (E-string) and the N-section cases $N = 2, 3, 4$ and the pseudo N-sections cases $N = 2', 3', 4'$ can be understood by specialisations of the mass parameters of the E-string partition, e.g. at

$b = 1$

$$\phi_1^{(N)}(\tau) = -\frac{1}{\eta(N\tau)^{12}} \sum_{i=2}^{4} \prod_{j=1}^{8} \theta_i \left(N\tau, v_j^{(N)} \cdot \tau \right) ,$$

by turning on Wilson loop parameters $\vec{v}^{\,(N)}$ on an S^1 compactification of the E-string given by

$$\begin{aligned} \vec{v}^{\,(1)} =& (0,0,0,0,0,0,0,0)\,, \\ \vec{v}^{\,(2)} =& (0,0,0,0,0,0,0,2) = \vec{\mu}_1\,, \quad \vec{v}^{\,(2')} = (0,0,0,0,0,0,1,1) = \vec{\mu}_8 \\ \vec{v}^{\,(3)} =& (0,0,0,1,1,1,1,4) = \vec{\mu}_5\,, \quad \vec{v}^{\,(3')} = (0,0,0,0,0,1,1,2) = \vec{\mu}_7 \\ \vec{v}^{\,(4)} =& (0,0,1,1,1,1,1,5) = \vec{\mu}_4\,, \quad \vec{v}^{\,(4')} = (0,0,0,0,1,1,1,3) = \vec{\mu}_6. \end{aligned} \tag{4}$$

These are specialisations of the E-string with restricted flavour symmetry classified by Eguchi and Sakai '02. The case $N = 2$ was used Kim^2 Kim,Lee, Park, Vafa in '14 to reconstruct the E-string elliptic genus from a conventional quiver description.

Nakajima's and Yoshioka's blow up equations

Origin of the Blow up equations: N=2 gauge theories in 4d and 5d (K-theoretic version).

NY used the equation to proof Nekrasov's 4d partition function defined by localisation on the gauge theory instanton moduli space

$$Z_{M_4}(\epsilon_1, \epsilon_2, \underline{a}, \underline{m}, q) = \sum_{n=0}^{\infty} q^n \int_{\mathcal{M}(r,n)} \mathbf{1}$$

$\mathcal{M}(r, n)$ is the framed moduli space of torsion free sheaves of rank r and $c_2(E) = n$ on $M_4 = \mathbb{C}^2$. $\mathbf{1}$ is an equivariant class w.r.t. to the torus action parametrized by ϵ_1, ϵ_2 and $\underline{a}$ on $\mathcal{M}(r,n)$. Here the ϵ_1, ϵ_2 acts on M_4 and is used to regularise the non-compactness of gauge theory instantons on non-compact spaces

The main goal of Nekrasov was to compute the prepotential on the Coulomb branch

$$F_0(\underline{a}, \underline{m}, \Lambda) = \lim_{\epsilon_1, \epsilon_2 \to 0} \frac{Z_N(\epsilon_1, \epsilon_2, \underline{a}, \underline{m}, q)}{\epsilon_1 \epsilon_2}$$

Geometric engineering relates the this prepotential and the refined higher genus terms to the topological string partition functions on local Calabi-Yau spaces.

The idea of NY's blow up equation is to blow up M_n in a point with an $\mathbb{P}^1$. It turns out that relate $Z_{\widehat{M_n}}$ relates Z_{M_n} in two ways:

- 1) by blowing down one recovers Z_{M_n},
- 2.) At the two fixpoints of the ϵ action on $\mathbb{P}^1$ the function Z_{M_n} appears with shifted localisation parameters.

This gives gives rise to the blow up equations for local

Calabi-Yau spaces Gu, Grassi '16, Huang, Sun,Wang '17

$$\sum_{\underline{n}\in\mathbb{Z}^{b_4^c}}(-1)^{|\underline{n}|}\widehat{Z}(\underline{t}+\epsilon_1\underline{R}_{\underline{n}},\epsilon_1,\epsilon_2-\epsilon_1)\times \\ \widehat{Z}(\underline{t}+\epsilon_2\underline{R}_{\underline{n}},\epsilon_1-\epsilon_2,\epsilon_2)$$

$$=\begin{cases}0, & \underline{r}\in\mathcal{S}_v, \\ \Lambda(\epsilon_1,\epsilon_2,\underline{m},\underline{r})Z(\underline{t},\epsilon_1,\epsilon_2) & \underline{r}\in\mathcal{S}_u.\end{cases}$$

Here

$$\underline{R}_{\underline{n}}=C\cdot\underline{n}+\underline{r}/2,$$

are vectors in $\mathbb{Z}^{b_4^c}$ that depend in the intersection on the local CY M.

$$C_{ij}=D_i\cdot C_j$$

with $[D_i]$, $i = 1, \ldots, b_4^c$, $[C_j]$, $i = 1, \ldots, b_2^c$ compact sub manifolds of M. $\hat{Z} = Z_{class} Z_{inst}$ and Z_{class} (and sometimes the genus zero invariants) are sufficient to solve the recursion provided by the blow up equation.

The 6d application leads to the elliptic blow up equations Haghighat, Gu, Klemm, Sun, Wang '18, '19 see Kaiwen Sun's Poster.

Conclusion:

- The topological string partitions function for all compact elliptic Calabi-Yau spaces can now be calculated to with the Jacobi-Form aproach.

- For the N-section case Z is expressible in terms of Jacobi forms of $\Gamma_1(N)$, $N = 2, 3, 4$.

- $K3$ fibrations with N-section are dual to the CHL compactifictions on $(K3 \times T2)/\mathbb{Z}_N$ which can be checked using BPS indices. Many new candidate elliptic CY can be identified.

- The local limit of N-section cases can be interpreted in $5d$. The 6d limit has intriguing non-local features.

- The Blow up equations for the local cases confirm the modular ansatz and are now the most efficient tool to get the BPS spectrum, for non-higgsable and higgsable theories.